A Concrete-Pictorial-Abstract Approach in Calculus Teaching

– A Little Help from Technology

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Introduction

- School students find calculus difficult (Toh 2021).
  - An over-emphasis of procedural knowledge over conceptual knowledge in schools
  - Over-use of symbols without emphasizing on the meaning
- Teachers also find calculus difficult (Toh, 2009; Toh et al., 2021).
  - They had difficulty with calculus when they were students
  - Undergraduate Calculus knowledge might not improve their understanding of school calculus.

Introduction

- Many terms used in calculus are incongruous to the terms used in the real world.

Introduction

- Usually, concepts are presented in algebraic modes, without a relational understanding of the underlying concepts.
- Students have difficulty with the pre-calculus concepts (functions and graphs).

Concrete-Pictorial-Abstract (CPA)

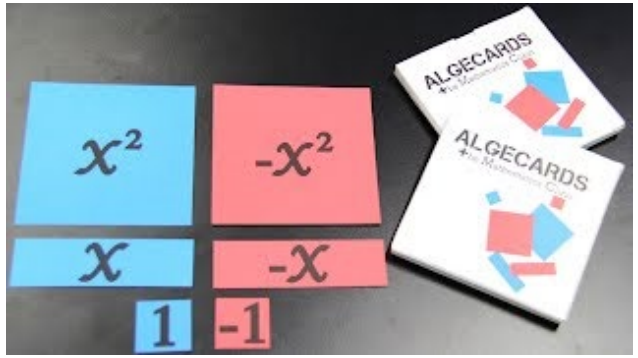
- Bruner (1966) on the cognitive development of children proposed three modes of representation
  - Enactive representation
  - Iconic representation
  - Symbolic representation
- The three representations explain how knowledge is stored and encoded in the memory

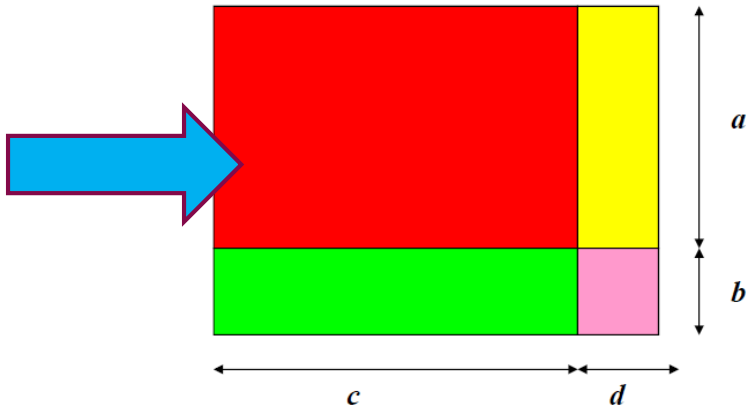
Concrete-Pictorial-Abstract (CPA)

- **This leads to the Concrete-Pictorial-Abstract (CPA) approach in teaching mathematics (esp. Algebra) since the early 1980s.**

Concrete-Pictorial-Abstract (CPA)

- Start with concrete manipulative
- Move with pictorial manipulative
- Develop a “mental manipulative” and translate it to procedure.





$(a + b)(c + d)$

$= ac + ad + bc + bd$

Simple Algebraic Expressions

Examples:

1) $3a + 6y$
   $= 3 (a + 2y)$

2) $fs + qr - fr - qs$
   $= fs - fr + qr - qs$
   $= f (s - r) + q (r - s)$
   $= (f - q) (s - r)$

Concrete **Pictorial** **Abstract**

Concrete Experience in (pre)Calculus

- General observation on how calculus textbooks are written: focus too much on the use of mathematical symbols and equations to discuss the procedures of calculus (e.g. differentiation and integration techniques, etc.).
- Relatively little time for exploring graphs and real-world context.
- Concrete: Need to re-interpret in the case of calculus.
- Concrete in its widened sense may not mean physical objects but could mean real life experience…

Concrete Experience in (pre)Calculus

- **Allow students the opportunity to experience calculus and pre-calculus concepts in the real world..**

Concrete Experience in (pre)Calculus

- Need students to develop experience with graphs. Creating given graphs using a motion sensor linked to a graphing calculator etc. Get students to experience creating different types of motion graphs by getting them to move.

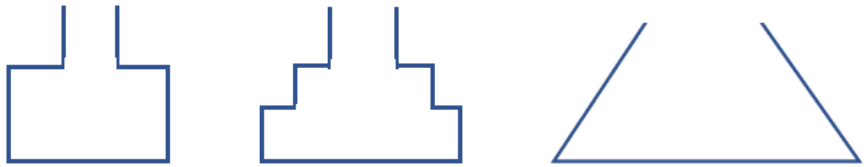
Concrete Experience in (pre)Calculus

- Graph: From **PLOTTING** to **SKETCHING**

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
h																					
(t, h)																					

Each of the following containers has water flowing into it at a constant rate. Sketch how the height of the water (h) changes with time (t).

(a) (b) (c)



Pictorial in Calculus

- Usually too little emphasis on the pictorial aspect.
- A study of the Singapore Secondary Calculus textbooks shows that the focus is on the computation rather than developing the concept image among the students.
- E.g. Instead of discussing the pictorial interpretation of calculus concepts (e.g. increasing / decreasing functions, maximum points), attention is quickly switched to the *algebraic* methods of finding these entities.

Pictorial in Calculus

- In a study conducted by Toh (2021), most students were not able to recognize the number of inflexion points in the given graph, although most of them should be able to perform the calculation to find the inflexion points.

Pictorial in Calculus

- Even though the students have been using graphing calculators and have learnt the various formulae for differentiation, they were unable to identify points on a graph for which the derivative does not exist (or that the function is not differentiable there). Reported in Toh (2021)

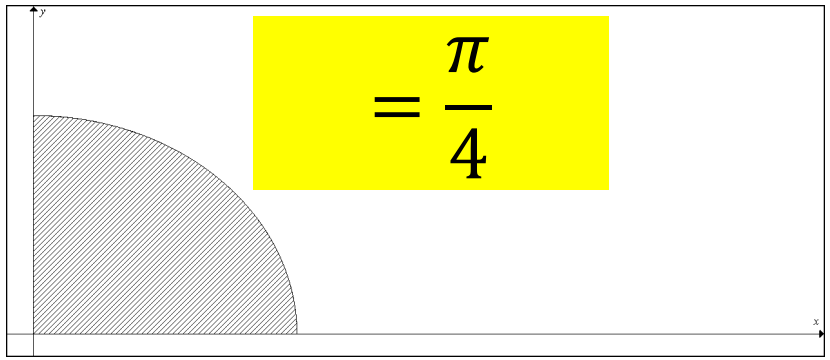
Pictorial in Calculus

- Local linearity versus Secant line method.

Pictorial in Calculus

$$\int_0^1 \sqrt{1-x^2}\,dx$$

Pictorial



$$= \frac{\pi}{4}$$

Use integration by parts:
Let *x* = sin *u*, then *dx* = sin *u* *du*
Then use substitution to reduce to a simple integral which can be evaluated directly.

Pictorial in Calculus

- Re-think a pictorial approach before an algorithmic approach

Evaluate (a) $\int_0^1 \sqrt{1-x^2}\,dx$ (b) $\int_0^{0.5} \sqrt{1-x^2}\,dx$

(c) $\int_{-3}^{3} \sin^{2011} x\,dx$

Pictorial in Calculus

- Technology does not just give us a quicker answer, but also change the way we think about mathematical concepts.
- Leads to the development of proceptual thinking – the ability to think of a mathematical entity both as a procedure and as a concept.

Abstract in Calculus

- **Advantage of a CPA approach in algebra: Students could always fall back when they are stuck with particular mathematics problem…**

Is $(a+b)^2 = a^2 + b^2$

OR

$(a+b)^2 = a^2 + b^2 + 2ab$

Abstract in Calculus

$$\int (2x+1)^{\frac{1}{2}} = \frac{(2x+1)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(2)} + C$$

$$\int (x^2+1)^{\frac{1}{2}} dx = \frac{(x^2+1)^{\frac{3}{2}}}{\frac{3}{2}(2x)} + C \text{ ???}$$

- Advantage of a CPA approach: Students could always fall back when they are stuck with particular mathematics problem…

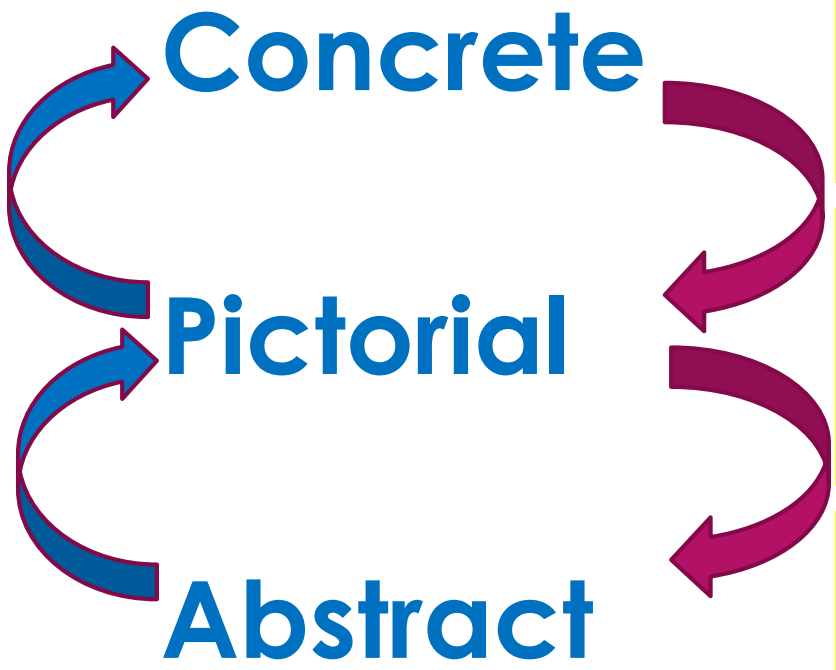
Abstract in Calculus

- **How to check that the working is incorrect?**
  - **Method 1: Integration as the reverse process of differentiation**
  - **Method 2: Graphical interpretation of the meaning of the definite integral and property of the graph.**

CPA approach to Calculus

- **Concrete part in pre-calculus**: building up the schema for the acquisition of calculus. Functions and graphs are a critical part of calculus.
- **Pictorial part**: building up the total concept image of the various calculus contents
- **Abstract part**: developing the proceptual thinking of an individual on the various concepts in calculus.

CPA approach to Calculus



Experiential Learning
Covariational Reasoning

Technology
Proceptual Thinking

Connections across
Abstract concepts

CPA approach to Calculus

- Many papers have been published on the teaching of calculus. This is an effort to pull together many research work that has been conducted.
- Many of the studies that involve the use of technology began as early as the 1980s.
- Re-think how technology today can play a part to enhance learning of calculus.